An Improved Algorithm for the Piecewise-Smooth Mumford and Shah Model in Image Segmentation

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An improved algorithm for the piecewise-smooth Mumford and Shah functional is presented. Compared to the previous work of Chan and Vese, and Choi et al., extensions of the key functions $u^{\pm}$ are replaced by updating the level set function based on an artificial image that is composed of the diffused image and the original image. The low convergence problem of the classical algorithm is efficiently solved in the proposed approach. The resulting algorithm has also been demonstrated by several cases.

1. INTRODUCTION

Image segmentation is one of the fundamental tasks of computer vision. Its goal is to partition a given image into regions that contain distinct objects. Active contours or "snakes" can be used to segment objects automatically. This framework has been used successfully by Kass et al. [1] to extract boundaries and edges. One potential problem with this approach is that the initial curve has to surround the objects to be detected, and interior contours can not be detected automatically. An algorithm to overcome this difficulty was first introduced by Osher and Sethian [2]. Chan and Vese [3] used a limiting version of Mumford and Shah (MS) [4] function, where the image was modeled as a piece-constant function. After that, they [5] extended the model to segment image using a particular multiphase level set formulation. However, the MS model in piecewise-constant case cannot detect objects successfully from noisy images. To overcome the drawback, Chan and Vese [6] showed how the piecewise-smooth MS segmentation problem could be solved using the level set method, and they had given the piecewise-smooth optimal approximations of a given image. Although the piecewise-smooth MS model works better, it requires the initial curve to be close to the boundaries, or the convergence of the curve to object boundary will be too slow, and for highly noisy images, it will almost collapse. Le and Vese [7] addressed the segmentation problem of images corrupted with additive or multiplicative noise by decomposing the images into three components, such as a piecewise-constant component,

a smooth component and noise. Motivated by the Chan and Vese approach, Lie et al. [8] proposed a variant of a PDE-based level set method, they solved the segmentation problem in a different way, that is, by introducing a piecewise-constant level set function. Instead of using the zero level of a function to represent the interface between subdomains, the interface is represented implicitly by the discontinuities of a level set function. Tsai et al. [9] addressed the problem of simultaneous image segmentation and smoothing by approaching the Mumford-Shah [4] paradigm from a curve evolution perspective. In particular, they defined a set of deformable contours as the boundaries between regions in an image where one could model the data via piecewise smooth functions and employ a gradient flow to evolve these contours.

In this paper, we propose a very efficient partial difference equation (PDE)-based algorithm to solve the low convergence problem of the piecewise-smooth MS segmentation functional. Different from the classical algorithms [6, 10], solution of the extensions of complementary functions u^{+} and u^{-} is replaced by updating the level set function on a compound image. The compound image can be regarded as an intermediate version of the original image so that the evolution of curves can be performed on it to adjust the pose and provide an additional drive force to speed up the convergence. In this paper, the piecewise-constant MS algorithm is applied to provide an additional drive force. So, the resulting algorithm has some advantages of being piecewise-constant MS model, such as faster speed of the evolution of

curves, better properties in edge preserving, and the evolution of curves being independent of the choice of the initial curve.

The rest of this paper is organized as follows. Section 2 describes the MS functional along with several variants, and introduces the notation. Section 3 describes the improved algorithm. Some results of the numerical experiments are given in Section 4, which is followed by conclusion in Section 5.

2. MUMFORD-SHAH MODEL

The Mumford-Shah model is a variational problem for approximating a given image by a piecewise smooth image of minimal complexity. Let $\Omega \in R^N$ be a bound domain with Lipschitz boundary, modeling the image domain. Let $u_0 : \Omega \to R$ represent a grayscale image. To find the segmentation Γ of u_0, Mumford-Shah piecewise smooth segmentation [4] is defined to carry out the following minimization:

$$
\inf_{u, \Gamma} E_{\text{MS}}(u, \Gamma \mid u_0)
$$
$$
= \int_\Omega (u - u_0)^2 dx + \mu \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \nu |\Gamma|, \quad (1)
$$

where μ and ν are positive parameters, u is the image intensity. It allows the segmented "objects" to have smoothly varying intensities. Chan and Vese [6] showed how the piecewise-smooth MS segmentation problem was solved using the level set method. In their model, two functions u^+ and u^- are introduced, such that

$$
u(x) = u^+(x)H(\phi(x)) + u^-(x)(1 - H(\phi(x))), \quad (2)
$$

where $H(z)$ is Heaviside function, and the authors regularized it as

$$
H(z) = \frac{1}{2}\left(1 + \frac{2}{\pi} \arctan\left(\frac{z}{\varepsilon}\right)\right). \quad (3)
$$

The two functions u^+ and u^- are assumed to be C^1 functions on $\phi \geq 0$ and $\phi < 0$, respectively, and with continuous derivatives up to all boundary points, that is, up to the boundary $\{\phi = 0\}$. Substituting this expression into (1), one can obtain

$$
\inf_{u^+, u^-, \Phi | u_0} E(u^+, u^-, \phi \mid u_0)
$$
$$
= \int_\Omega |u^+ - u_0|^2 H(\phi) dx + \int_\Omega |u^- - u_0|^2 (1 - H(\phi)) dx
$$
$$
+ \mu \int_\Omega |\nabla u^+|^2 H(\phi) dx + \mu \int_\Omega |\nabla u^-|^2 (1 - H(\phi)) dx
$$
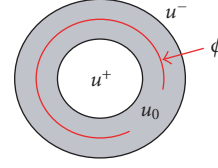$$
+ \nu \int_\Omega |\nabla H(\phi)|. \quad (4)
$$

FIGURE 1: An image that is composed of u^+, u^-, and the original image u_0.

Then with ϕ fixed, (4) leads to the two Euler-Lagrange equations for u^+ and u^- written as

$$
u^+ - u_0 = \mu \Delta u^+, \qquad \{(x) : \phi(x, t) > 0\},
$$
$$
\frac{\partial u^+}{\partial \vec{n}} = 0, \qquad \{(x) : \phi(x, t) = 0\} \cup \partial \Omega,
$$
$$
u^- - u_0 = \mu \Delta u^-, \qquad \{(x) : \phi(x, t) < 0\}, \quad (5)
$$
$$
\frac{\partial u^-}{\partial \vec{n}} = 0, \qquad \{(x) : \phi(x, t) = 0\} \cup \partial \Omega.
$$

Notice that u^+ and u^- act as denoising operators on the homogeneous regions only. No smoothing is done across the boundary $\{\phi = 0\}$, which is very important in image analysis.

Now, keeping u^+ and u^- fixed, and minimizing $E_{\text{MS}}(u^+, u^-, \phi \mid u_0)$ with respect to the function ϕ, one can obtain the motion of the zero level set as the following:

$$
\frac{\partial \phi}{\partial t} = \delta(\phi)\left(\nu \nabla\left(\frac{\nabla \phi}{|\nabla \phi|}\right) - |u^+ - u_0|^2 - \mu |\nabla u^+|^2\right.
$$
$$
\left. + |u^- - u_0|^2 + \mu |\nabla u^-|^2\right), \quad (6)
$$

where the delta function is defined as the derivative of the Heaviside function:

$$
\delta(z) = \frac{1}{\pi}\left(\frac{\varepsilon}{\varepsilon^2 + z^2}\right). \quad (7)
$$

The above (6) with some initial guesses $\phi(t = 0, x)$ is actually computed at least near a narrow band of the zero level set. As a result, computationally, one has to continuously extend both u^+ and u^- from their original domain $\{\pm \phi > 0\}$ to a suitable neighborhood of the zero level set $\{\phi = 0\}$. Although u^+ and u^- can be easily obtained by solving Euler-Lagrange equations (5), the extensions of u^+ and u^- are very difficult to be solved. we have to solve the following degenerate elliptic linear equations:

$$
u_t^+ = \nabla^2 u^+ (\vec{N}, \vec{N}), \quad \{\phi < 0\},
$$
$$
\frac{\partial u^+}{\partial \vec{n}} = 0,
$$
$$
u_t^- = \nabla^2 u^- (\vec{N}, \vec{N}), \quad \{\phi > 0\}, \quad (8)
$$
$$
\frac{\partial u^-}{\partial \vec{n}} = 0.
$$

Chan and Vese [6] had pointed out three possible ways to solve the problem, but all of them were difficult to carry out in practice. So in this paper, a new strategy is proposed to solve the problem. It will be described in following sections.
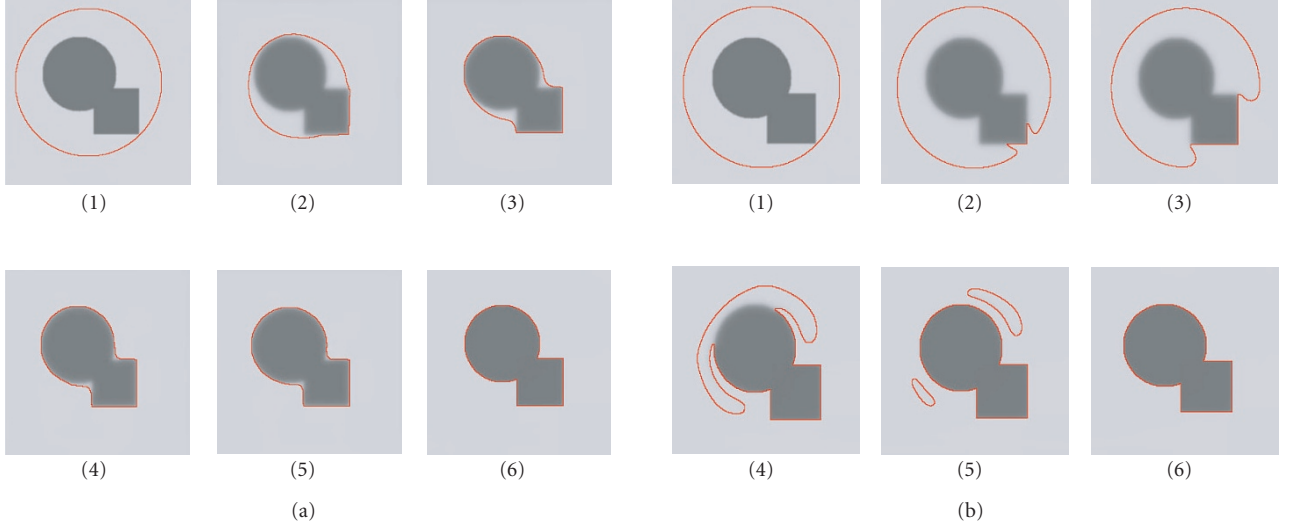
FIGURE 2: Segmenting an artificial image with furry edges: (a) by the improved algorithm with 54 iterations and (b) by the original algorithm with 725 iterations.
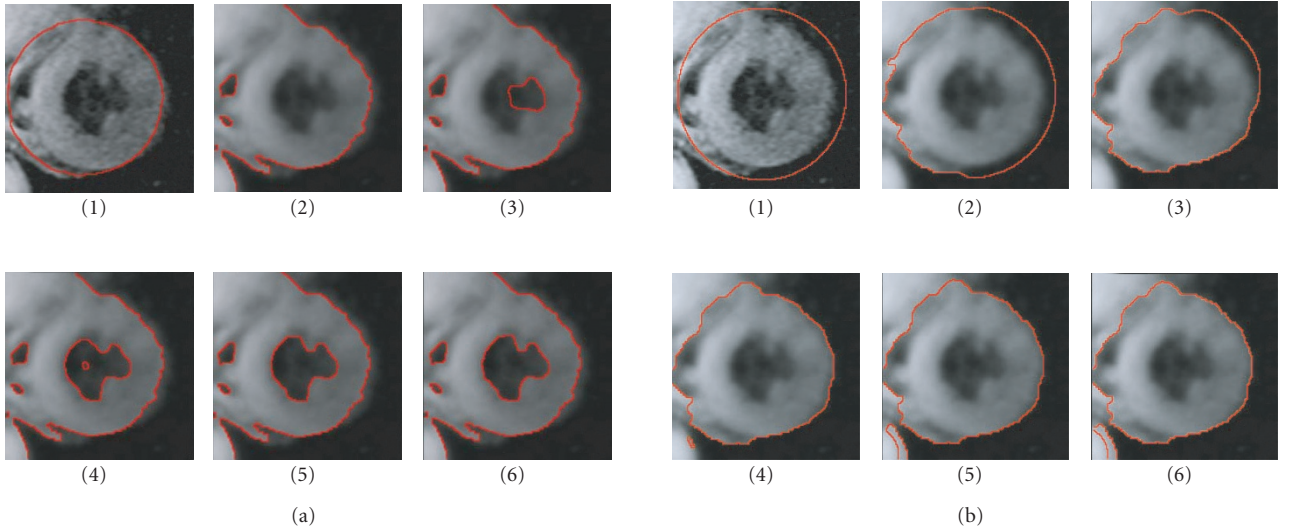

FIGURE 3: Segmenting a heart image: (a) by the improved algorithm with 201 iterations and (b) by the original algorithm with 1315 iterations.

3. PROPOSED NEW ALGORITHM

To solve the two Euler-Lagrange equations in (5), a new strategy is proposed to drive directly the evolution of curves on a compound image by an external force to replace the solution of extensions of $u^{\pm}$. Since the evolution of curves coupled with diffusion in the piecewise-smooth MS model, the resulting image might become very homogeneous in certain iterations. To drive the evolution of curves, a lot of approaches, in theory, could be applied for this purpose. Note that the piecewise-constant MS functional works better for homogeneous regions and, in theory, robust, hence it is the best appropriate candidate to be used for the purpose. As known in previous sections, to keep the evolution

of curves, both u^{+} and u^{-} have to be continuous extended from their original domain $\{\pm\phi > 0\}$ to a suitable neighborhood of the zero level set $\{\phi = 0\}$. Considering that $u^{\pm}$ in (5) act as a denoising operator on homogeneous regions outside or inside the boundaries $\{\phi = 0\}$, respectively, therefore a smoothing diffused image can be obtained by calculating the union of u^{+} on $\{\phi > 0\}$ and u^{-} on $\{\phi < 0\}$. Based on this idea, one can directly develop the level set function ϕ on the diffused image instead of the extensions of u^{+} and u^{-}. Because the smoothing operator will blur the boundaries of objects, the contours or edges of the diffused images will become more and more blurry as the evolution of curves progresses. To overcome the drawback, a narrowband is defined on the diffused image and bounded on either side by two
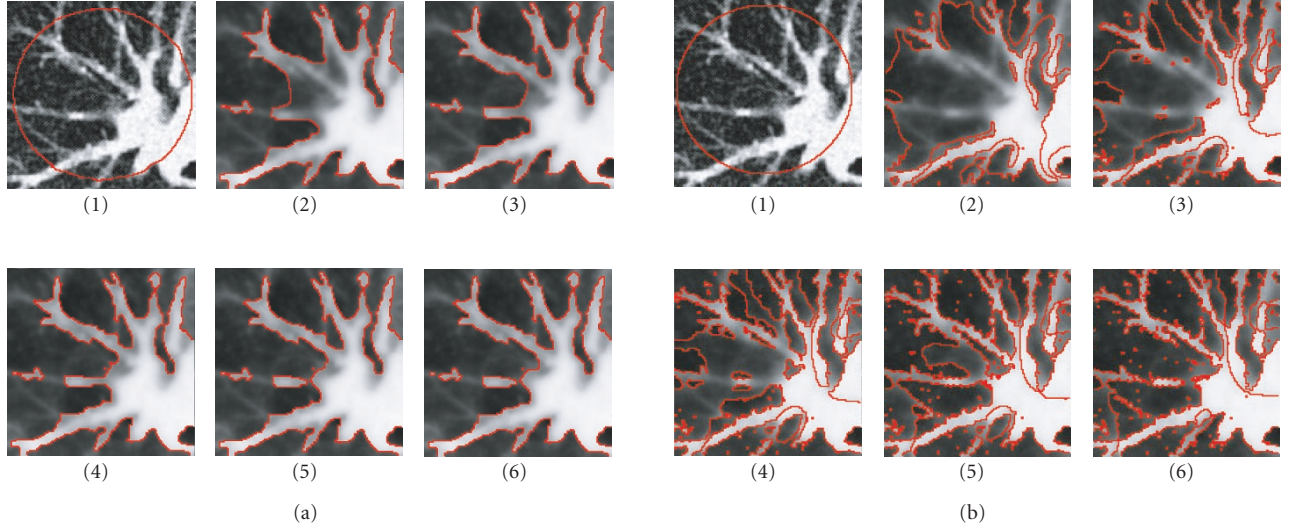
FIGURE 4: Segmenting of a blood vessel image: (a) by the improved algorithm with 610 iterations and (b) by the original algorithm with 2320 iterations.

curves which are a distance τ apart, that is, the two curves are level sets $\{\phi = \pm\tau/2\}$. Here the pixel points that fall within the narrowband are obtained from the original image. Moreover a compound image is composed of u^+, u^-, and the region of narrowband as shown in Figure 1. Let τ denote the width of the narrowband, and the compound image ξ can be represented as follows:

$$\xi(x) = u^+\left\{\phi > \frac{\tau}{2}\right\} \cup u^-\left\{\phi < -\frac{\tau}{2}\right\} \cup u_0\left\{|\phi| < \frac{\tau}{2}\right\}. \quad (9)$$

By updating the level set function $\{\phi = 0\}$ on the compound image ξ by the piecewise-constant MS functional at each time steps, computation of u^+ and u^- will be performed alternatively based on the new location of the level set function. Consider that the singularity may happen in flat regions while $|\nabla\phi| = 0$, thus a small parameter $\varepsilon > 0$ is applied. The algorithm can be outlined as follows.

(1) Initialize the distance functions $\phi_{i,j}^0$ (the initial curve), set $n = 0$, $u_{i,j}^{0,+} = u_{i,j}^{0,-} = u_0$, and $\tau = 1.5$ for each $n > 0$ until steady state.

(2) Compute $u_{i,j}^{n,+}$ and $u_{i,j}^{n,-}$ with (5).

(3) Compute the image ξ as the current "original" image $\hat{u}_0$:

$$\hat{u}_0 = u^+\left\{\phi > \frac{\tau}{2}\right\} \cup u^-\left\{\phi < -\frac{\tau}{2}\right\} \cup u_0\left\{|\phi| < \frac{\tau}{2}\right\}. \quad (10)$$

(4) Compute $\hat{\phi}_{i,j}^{n+1}$ based on the piecewise-constant MS functional with one time step, as the following:

$$\begin{aligned}
\hat{\phi}_{i,j}^{n+1} = \frac{1}{C}\{\phi_{i,j}^n &+ m_1(C_1\phi_{i+1,j}^n + C_2\phi_{i-1,j}^n \\
&+ C_3\phi_{i,j+1}^n + C_4\phi_{i,j-1}^n) + \Delta t\delta_\varepsilon(\phi) \\
&\times [-\nu(\hat{u}_0 - c_1)^2 + (\hat{u}_0 - c_2)^2]\},
\end{aligned} \quad (11)$$

where

$$c_1 = \frac{\int_\Omega \hat{u}_0 H_\varepsilon(\phi)\,dx}{\int_\Omega H_\varepsilon(\phi)\,dx}, \qquad c_2 = \frac{\int_\Omega \hat{u}_0(1 - H_\varepsilon(\phi))\,dx}{\int_\Omega (1 - H_\varepsilon(\phi))\,dx},$$

$$C_1 = \frac{1}{\sqrt{\varepsilon^2 + ((\phi_{i+1,j}^n - \phi_{i,j}^n)/h)^2 + ((\phi_{i,j+1}^n - \phi_{i,j-1}^n)/2h)^2}},$$

$$C_2 = \frac{1}{\sqrt{\varepsilon^2 + ((\phi_{i,j}^n - \phi_{i-1,j}^n)/h)^2 + ((\phi_{i-1,j+1}^n - \phi_{i-1,j-1}^n)/2h)^2}},$$

$$C_3 = \frac{1}{\sqrt{\varepsilon^2 + ((\phi_{i,j+1}^n - \phi_{i,j}^n)/h)^2 + ((\phi_{i+1,j}^n - \phi_{i-1,j}^n/2h))^2}},$$

$$C_4 = \frac{1}{\sqrt{\varepsilon^2 + ((\phi_{i,j}^n - \phi_{i,j-1}^n)/h)^2 + ((\phi_{i+1,j-1}^n - \phi_{i-1,j-1}^n)/2h)^2}},$$

$$m_1 = \frac{\Delta t}{h^2}\delta_\varepsilon(\phi)\nu,$$

$$C = 1 + m_1(C_1 + C_2 + C_3 + C_4). \quad (12)$$

(5) Set $\phi_{i,j}^n = \hat{\phi}_{i,j}^{n+1}$ and compute $\phi_{i,j}^{n+1}$ using (6).

4. NUMERICAL EXPERIMENTS

In this section, we present the results of numerical experiments that were obtained using the improved algorithm. All tests are performed on personal computer (1.7 GHz CPU with 512 MB of RAM) under the MS-Windows operating system. The algorithm has been implemented in the Visual C++ 6.0. For comparison we have used the following parameter values with the time step $\Delta t = 0.1$, space steps $h = \Delta x = \Delta y = 1$, $\mu = 1.0$, and $\nu = 0.0305 * 255^2$ in

our experiment, which are the same as those in [11]. The width of narrowband τ, here $\tau = 1.5$, is used to create the compound image, which imposes upper and lower limits to the level set function. An appropriate band width cannot only avoid detecting some extra contours which do not correspond to physical edges but can also make the algorithm more computationally efficient. When $\tau = 0$ or $\tau > 5$ the convergence of the curve to object boundary will become too slow, although the algorithm still stops at the correct boundaries of objects. By numerical experiment we found that better results could be obtained with $\tau = 1.0 \sim 3.0$ for general images. Figure 2 demonstrates an advantage of the proposed approach in speeding up convergence. Only 54 iterations were necessary to segment the artificial image with furry edges (Figure 2(a)) by the improved algorithm. Figure 2(b) shows the results of segmenting the same image by original algorithm with 725 iterations taken to reach an essentially state. In Figure 3 we show a heart image where the classical algorithm fails to stop at the correct boundaries, thus, our algorithm can do better on this kind of image. Figure 4 demonstrates another advantage of the improved algorithm in preventing nonphysical components on the noisy image (Figure 4(a)). Figure 4(b) also shows the results of segmenting the same image by the original algorithm, and considerable nonphysical components were introduced.

5. CONCLUSION

In this paper, we describe an efficient and reliable improved algorithm for the piecewise-smooth Mumford-Shah segmentation problem with edge preserving. Unlike the classic algorithms [6, 10], computing the extensions of functions u^+ and u^- is replaced by directly updating the level set function on a compound image using the piecewise-constant MS method. We have tested the proposed algorithm by some medical images and other images, and proved that it is more efficient, and converges faster than classical one; moreover, it can work better on some highly noisy images that the classical algorithms fail to convergence. Like the Chan-Vese approach, however, there are a few parameters to be determined carefully for better segmentation results. The difficulties are how to determine the parameters reasonably, which need to be researched further.

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